Spherical Monofractal and Multifractal Random Fields with Cosmological Applications

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AustMS 2020: 64th Annual Meeting of the Australian Mathematical Society
December 9, 2020

The talk is based on joint results with Professors Nikolai Leonenko (Cardiff University, UK) and Andriy Olenko (La Trobe University, Australia).

Overview

Cosmological background

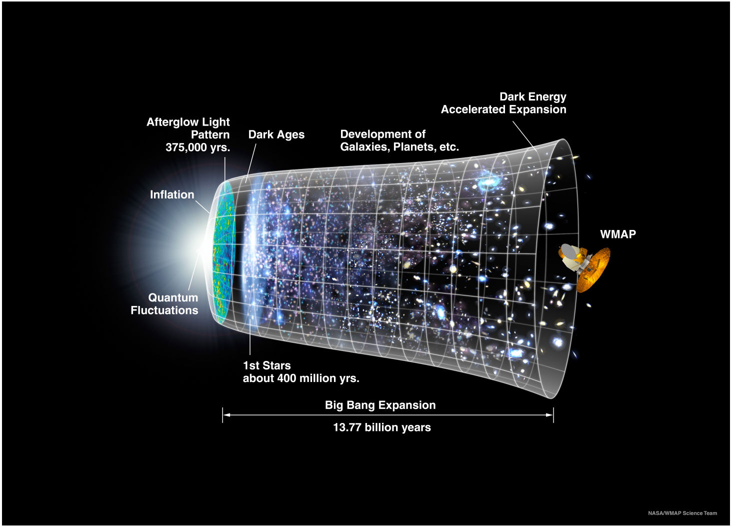

Image credit: NASA / WMAP Science Team

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Cosmological background

- The Cosmic Microwave Background (CMB) is the radiation from the universe since 380,000 years from the Big Bang.
- In Big Bang cosmology, the CMB is an electromagnetic radiation residue from its earliest stage.
- The CMB depicts variations which correspond to different regions and represents the roots for all future formation including the solar system, stars and galaxies.
- The unforeseen discovery of the CMB was done by Arno Penzias and Robert Wilson who were American radio astronomers.

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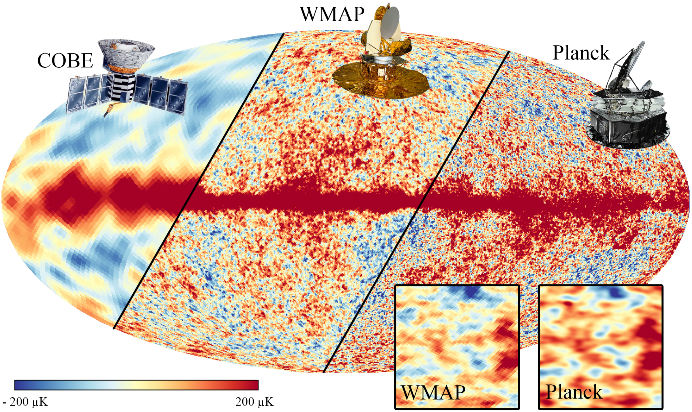
Missions



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Planck mission

- In the year 2009, the European Space Agency launched the mission Planck to study the CMB thoroughly.
- The frequency range captured by the Planck mission is much wider and its sensitivity is higher than that of the previous missions, Cosmic Background Explorer (COBE) and Wilkinson Microwave Anisotropy Probe (WMAP).
- One of the aims of the Planck mission was to verify the standard model of cosmology using this achieved greater resolution and to find out fluctuations from the specified standard model of cosmology.
- Current CMB data are at 5 arc-minutes resolution on the sphere.
- Contains 50,331,648 data collected by Planck mission.

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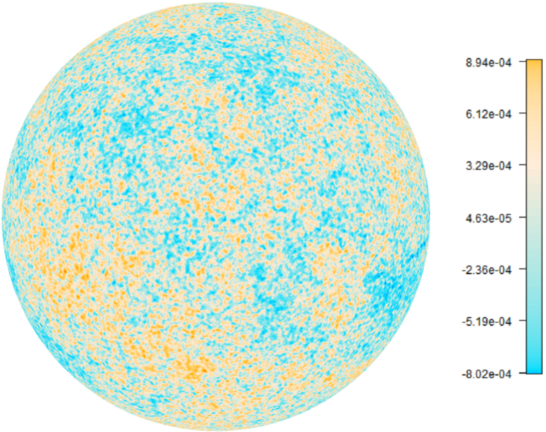
What does CMB data look like?

- Obtained using the **"rcosmo"** package.



Random fields on a sphere

- The spherical surface in $\mathbb{R}^3$ with a given radius $r > 0$ is $s_2(r) = \{x \in \mathbb{R}^3 : \|x\| = r\}$, with the corresponding Lebesgue measure on the sphere $\sigma_r(d\theta \cdot d\varphi) = r^2 \sin\theta d\theta d\varphi$, $(\theta, \varphi) \in s_2(1)$.

- **Statistical model:** CMB can be viewed as a single realization of a random field on a sphere.

- A spherical random field $T = \{T(r, \theta, \varphi) : 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi, r > 0\}$ is a random function, which is defined on the sphere $s_2(r)$.

- We deal with a spherical real-valued mean-square continuous random field T with a constant mean and finite second order moments.

Random fields on a sphere

Definition 1

A real-valued second order random field $T(x), x \in s_2(r)$, with $E[T(x)] = 0$ is isotropic if $E[T(x_1)T(x_2)] = B(\cos\Theta)$, $x_1, x_2 \in s_2(r)$, depends only on the angular distance Θ between x_1 and x_2.

An isotropic spherical random field on $s_2(r)$ can be expanded in a Laplace series in the mean-square sense.

$$T(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^m(\theta, \varphi) a_l^m(r), \quad (1)$$

where $\{Y_l^m(\theta, \varphi)\}$ represents the spherical harmonics defined as

$$Y_l^m(\theta, \varphi) = c_l^m \exp(im\varphi) P_l^m(\cos\theta), \quad l = 0, 1, ..., \ m = 0, \pm 1, ..., \pm l,$$

with

$$c_l^m = (-1)^m \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2}.$$

Multifractality

- Multifractal structures are typical in nature and multifractal models have been extensively used in the fields of geophysics, genomics, image modelling, finance, meteorology, etc.
- The concept of multifractality initially emerged in the context of measures.
- B. Mandelbrot showed the significance of scaling relations in the setting of turbulence modelling.
- A multifractal pattern is a type of a fractal pattern that scales with multiple scaling rules in contrast to monofractals that have only scaling rule.

Main notations and definitions

Definition 2

A stochastic process $\{X(t), t \in \mathbb{R}^d\}$ is self-similar if for any non-random constant $a > 0$, there exists non-random constant $b > 0$ such that $\{X(at)\} \overset{d}{=} \{bX(t)\}$.

Definition 3

A stochastic process $X(t)$ is multifractal if it holds $\{X(ct)\} \overset{d}{=} \{M(c)X(t)\}$, where $M(c)$ is a random variable independent of $X(t)$ for every $c > 0$ and the distribution of $M(c)$ does not depend on t.

Definition 4

A stochastic process $X(t)$ is multifractal if there exist non-random functions $c(q)$ and $\tau(q)$ such that for all $t, s \in \mathcal{T}, q \in \mathcal{Q}$,

$$E|X(t) - X(s)|^q = c(q)|t - s|^{\tau(q)},$$

where $\mathcal{T}$ and $\mathcal{Q}$ are intervals on the real line with positive length and $0 \in \mathcal{T}$.

Main notations and definitions

Condition 1

Let the random field $\tilde{\Lambda}(x)$, $x \in s_2(1)$, *satisfy*

$$E\tilde{\Lambda}(x) = 1, \quad Var\tilde{\Lambda}(x) = \sigma^2_{\tilde{\Lambda}} < \infty, \quad \tilde{\Lambda}(x) > 0,$$

$$Cov(\tilde{\Lambda}(\theta, \varphi), \tilde{\Lambda}(\theta^{'}, \varphi^{'})) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) C_l P_l(\cos\theta), \quad \sum_{l=0}^{\infty} (2l+1) C_l < \infty.$$

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Rényi function

- Rényi function which is also known as the deterministic partition function plays a key role in multifractal analysis.
- The Rényi function computes how the measure/mass/intensity on a surface varies with the resolution or the block size of an image.
- There are several scenarios where the Rényi function was computed for the one-dimensional case and time-series.
- However, there are very few multidimensional models where it is given in an explicit form.
- Leonenko and Shieh (2013) computed the Rényi function for three models of the multifractal random fields and showed some major schemes for the Rényi function that reveal the multifractality of homogeneous and isotropic data.

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Main notations and definitions

Let $\tilde{\Lambda}^{(i)}(x), x \in s_2(1),\ i = 0, 1, 2, ...,$ be a sequence of independent copies of the field $\tilde{\Lambda}(\cdot)$.

Define the finite product fields on $s_2(1)$ by

$$\tilde{\Lambda}_k(x) = \prod_{i=0}^{k} \tilde{\Lambda}^{(i)}(b^i \times x), \quad k = 1, 2, ...$$

Let us introduce the random measure $\mu_k(\cdot)$ on the Borel σ-algebra $\mathcal{B}$ of $s_2(1)$ as

$$\mu_k(A) = \int_A \tilde{\Lambda}_k(y) dy,\ k = 0, 1, 2, ...,\ A \in \mathcal{B}.$$

We denote by $\mu_k \xrightarrow{d} \mu,\ k \to \infty$, the weak convergence of the measures μ_k to some non-degenerate measure μ. It means that for all continuous functions $g(y), y \in s_2(1)$,

$$\int_{s_2(1)} g(y)\mu_k(dy) \to \int_{s_2(1)} g(y)\mu(dy), \quad k \to \infty.$$

The Rényi function

- The Rényi function of the random measure μ defined on $s_2(1)$ is defined as

$$T(q) = \liminf_{m\to\infty} \frac{\log_2 E \sum_l \mu(S_l^{(m)})^q}{\log_2 |S_l^{(m)}|}, \tag{2}$$

where $\{S_l^{(m)}, l = 0, 1, ..., 2^m - 1\}$ is the mesh constructed by m^{th} level dyadic decomposition of the spherical surface of $s_2(1)$.

- The multifractal or singularity spectrum is defined via the Legendre transform as

$$f(h) = \inf_q (hq - T(q)). \tag{3}$$

and is used to describe local fractal dimensions of random fields.

Main notations and definitions

Theorem 1

Suppose that Condition 1 holds and the isotropic random field $\tilde{\Lambda}(\cdot)$ is the restriction to the sphere $s_2(1)$ of the HIRF $\Lambda(x), x \in \mathbb{R}^3$, with the correlation function $\rho_\Lambda(\|x-y\|) = \rho(r)$.

- (i) *Assume that the correlation function $\rho_\Lambda(\|x-y\|) = \rho(r)$ of the field $\Lambda(\cdot)$ satisfies the following condition*

$$|\rho_\Lambda(r)| \leq Ce^{-\gamma r}, \ r > 0, \tag{4}$$

  for some positive constants C and γ. Then, for the scaling factor $b > \sqrt[3]{1+\sigma_\Lambda^2}$, the measures $\mu_k \xrightarrow{d} \mu$, $k \to \infty$, on S^2.

- (ii) *Then, the Rényi function $T(q)$ of the limit measure μ on $s_2(1)$ is given by*

$$T(q) = q - 1 - \frac{1}{2}\log_b E\Lambda^q(t), \quad q \in Q.$$

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Conditions on measure μ

- Previously, the random measure μ was defined as a weak limit of the measures μ_k.
- Therefore, it would be difficult to check moment conditions on μ as its probability distribution is not explicitly known.
- Here, we provide sufficient conditions on the scaling factor b and the variance σ^2_Λ that guarantee $E\mu^q(B^3) < \infty$.

Theorem 2

Let the mother field $\Lambda(x) > 0$, $x \in \mathbb{R}^3$, satisfy the conditions

$$E\Lambda(x) = 1, \quad Var\Lambda(x) = \sigma^2_\Lambda < +\infty, \quad Cov(\Lambda(x), \Lambda(y)) = \sigma^2_\Lambda \rho_\Lambda(\|x - y\|),$$

$$|\rho_\Lambda(\tau)| \leq Ce^{-\gamma\tau}, \ \tau > 0,$$

and the scaling factor $b > \max(\sqrt[3]{1 + \sigma^2_\Lambda}, \ e^{\frac{\sigma^2_\Lambda C}{3}})$.
Then the measures $\mu_k \xrightarrow{d} \mu$, $k \to \infty$, and $E\mu^q(B^3) < +\infty$, for $q \in [1, 2]$.

Conditions on measure μ

- The most known results are proved only for q in [1,2].

Remark 1

The conditions on b and σ_λ^2 that guarantee $E\mu_k^4(B^3) < +\infty$ are also sufficient for $E\mu^q(B^3) < +\infty$, $q \in [1,4]$. Then, it follows from

$$E\mu_k^4(B^3) = \int_{B^3}\int_{B^3}\int_{B^3}\int_{B^3} \prod_{i=0}^{k} E\left(\prod_{j=1}^{4} \Lambda^{(i)}(y_i b^i)\right) \prod_{j=1}^{4} dy_i,$$

that one can impose some additional assumptions on the fourth order moments $E(\prod_{j=1}^{4} \Lambda(y_j b^i))$ or cumulants of the mother field $\Lambda(\cdot)$.

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Known results about the Rényi function

- For the random fields on the sphere, there are three models in the literature where the Rényi function is known explicitly.
- They are log-normal model, log-gamma model and log-negative-inverse-gamma model.
- These results were obtained for exponential type spherical random fields.

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Models based on power transformations of Gaussian fields

Model 4: Let $\Lambda(x) = Y^2(x)$, where $Y(x)$, $x \in \mathbb{R}^3$, is a zero-mean unit variance Gaussian HIRF with a covariance function $\rho_Y(\tau)$, $\tau \geq 0$.

Theorem 5

Suppose that for Model 4, the correlation function of $Y(x)$ satisfies $|\rho_Y(r)| \leq Ce^{-\gamma r}, r > 0, \gamma > 0$, and $b > \max(\sqrt[3]{1+\sigma_\lambda^2}, e^{\sigma_\Lambda^2 C/3})$.
Then the measures $\mu_k \xrightarrow{d} \mu$, $k \to \infty$, and the corresponding Rényi function is equal to

$$T(q) = q - 1 - \frac{1}{2}\log_b\left(\frac{2^q\Gamma(q+\frac{1}{2})}{\sqrt{\pi}}\right), \; q \in [1,2].$$

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Models based on power transformations of Gaussian fields

Model 5: Let $\Lambda(x) = Y^{2k}(x)$, $k \in \mathbb{N}$, where $Y(x)$, $x \in \mathbb{R}^3$, is a zero-mean Gaussian HIRF with the variance $\sigma^2 = \left(\frac{\sqrt{\pi}}{2^k \Gamma(k+\frac{1}{2})}\right)^{-\frac{1}{k}}$ and a covariance function $\rho_Y(r), r \geq 0$.

Theorem 6

Suppose that for Model 5 the correlation function of $Y(x)$ satisfies $|\rho_Y(r)| \leq Ce^{-\gamma r}$, $r > 0$, $\gamma > 0$, and $b > \max\left(\sqrt[3]{1+\sigma_\Lambda^2},\ e^{\frac{\sigma_\Lambda^2 C}{3}}\right)$.

Then the measures $\mu_k \xrightarrow{d} \mu$, $k \to \infty$, and the Rényi function is given by

$$T(q) = q - 1 - \frac{1}{2}\log_b EY^{2kq}(x) = q - 1 - \frac{1}{2}\log_b\left(\frac{2^{kq}\Gamma(kq+\frac{1}{2})}{\sqrt{\pi}}\right).$$

for $q \in [1, 2]$.

Models based on power transformations of Gaussian fields

Model 6: Let $\Lambda(x) = \frac{2}{k} Y(x)$, $k \in \mathbb{N}$, where $Y(x) \sim \chi^2(k)$, and the HIRF field $Y(x)$, $x \in \mathbb{R}^3$, has a covariance function $\rho_Y(r)$, $r \geq 0$.

Theorem 7

Suppose that the correlation function in Model 6 satisfies the inequality $|\rho_Y(r)| \leq Ce^{-\gamma r}$, $r > 0$, $\gamma > 0$, *and* $b > \max\left(\sqrt[3]{1+\sigma_\Lambda^2},\ e^{\frac{\sigma_\Lambda^2 C}{3}}\right)$.

Then the measures $\mu_k \xrightarrow{d} \mu$, $k \to \infty$, *and for* $q \in [1,2]$ *the Rényi function is equal to*

$$T(q) = q\left(1 - \frac{1}{2}\log_b\left(\frac{2}{k}\right)\right) - 1 - \frac{1}{2}\log_b\left(2^q \frac{\Gamma(q+\frac{k}{2})}{\Gamma(\frac{k}{2})}\right).$$

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Computing multifractal spectra for the models

Let $\alpha(q)$ denote the q^{th} order singularity exponent and be defined by

$$\alpha(q) = \frac{d}{dq} T(q).$$

Then the multifractal spectrum defined by (3) can be expressed as a function of α by

$$f(\alpha(q)) = q \cdot \alpha(q) - T(q).$$

Multifractal spectra for model 4:

$$f(\alpha(q)) = 1 + \frac{1}{2} \log_b \left(\frac{\Gamma(q + \frac{1}{2})}{\sqrt{\pi}} \right) - \frac{q\psi(q + \frac{1}{2})}{2 \ln 2}$$

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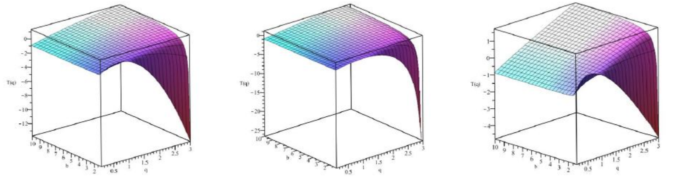
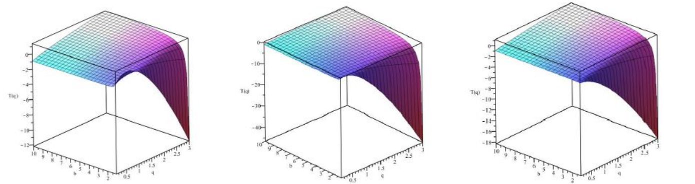
Dependence of the Rényi function on the parameter b



(a) Rényi functions of Model 1 (b) Rényi functions of Model 2 (c) Rényi functions of Model 3

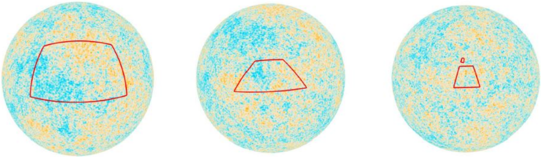


(d) Rényi functions of Model 4 (e) Rényi functions of Model 5 (f) Rényi functions of Model 6

Figure 1: Dependence of the Rényi function on the parameter b.

Computing the Rényi function for CMB data

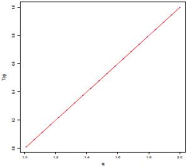
- Empirical Rényi functions were calculated for real cosmological data obtained from the NASA/IPAC Infrared Science Archive.
- Extensive numerical studies were conducted for different windows in various sky locations.
- For the CMB data analysis, we use $\mu(S_l^m)$ which equals the cumulative CMB intensity over S_l^m.
- The statistical estimator $\widehat{T(q)}$ is obtained using the equation (2) and for large values of m.



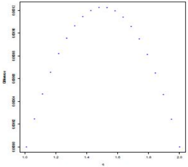
(a) Large window (b) Medium window (c) Small and very small windows

Figure 2: Different sky windows of CMB data.

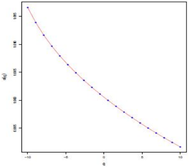
Computing the Rényi function for CMB data



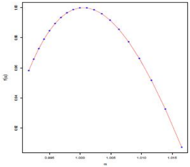
(a) Sample Rényi function versus linear function



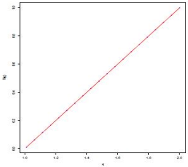
(b) Difference of sample Rényi function and linear function



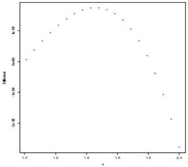
(c) $\alpha(q)$ versus q



(d) $f(\alpha)$ versus α



(e) Sample Rényi function with the fitted Log-Normal model



(f) Difference between sample Rényi function and the fitted Log-Normal model

Figure 3: Whole sky data analysis.

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Computing the Rényi function for CMB data

- First, the empirical Rényi function was computed for the whole sky and each of the previous models were fitted to it.
- The Rényi functions, multifractal spectra, similar analysis and plots were produced for different window sizes of the CMB unit sphere.
- Large, medium, small and very small window sizes with areas 1.231, 0.4056, 0.0596 and 0.0017 were selected.
- The empirical Rényi function was computed for small windows located at different places of the sky sphere such as near the pole, near the equator and other places of the sphere.
- It is clear from the analysis that the departure from a linear behaviour is not substantial.
- All these plots confirm only very small multifractality of the CMB data.
- Although different window sizes of the sphere were investigated, there's not that much of evidence to suggest that we have substantial multifractality.

Computing the Rényi function for CMB data

Observation window	$[\alpha_{\min}, \alpha_{\max}]$	$\alpha_{\max}$ - $\alpha_{\min}$	a	RMSE
Whole Sky	[0.9916, 1.0165]	0.024917	0.000513	$1.3602 \cdot 10^{-6}$
Large	[0.9908, 1.0167]	0.025846	0.000555	$1.3590 \cdot 10^{-6}$
Medium	[0.9893, 1.0159]	0.026620	0.000629	$1.1033 \cdot 10^{-6}$
Small	[0.9867, 1.0170]	0.030219	0.000745	$7.9095 \cdot 10^{-7}$
Very Small	[0.9842, 1.0543]	0.070150	0.001500	$1.3949 \cdot 10^{-5}$

Figure 4: Analysis of different sky windows data with Model 1.

- For the log-normal model, the simple linear regression approach was used whereas for the other models, the non-linear regression approach was applied.
- The values of the parameter $a = \frac{\sigma_Y^2}{4 \ln b}$ resulting in the form $T(q) = a(-q^2 + q) + q - 1$ and the root mean square error for deviations of log-normal model from the empirical Rényi function are given in the above table.
- The results also confirm that multifractality is very small as for all observation windows, a is almost zero and $\alpha_{\max} - \alpha_{\min}$ is very small.

Conclusions

- This study investigates the multifractal behaviour of spherical random fields and some applications to cosmological data from the Planck mission.
- The aim of this study is to introduce several multifractal models for random fields on a sphere and to propose simpler models where the Rényi function can be computed explicitly.
- All the Rényi functions for the specified models exhibit either parabolic or approximately linear behaviours.
- We present the Rényi function computations for different CMB sky windows located at different places of the sphere.
- All the specified models fit to the actual CMB data.
- The analysis suggests that there may exist a very minor multifractality of the data.

Future work

- Develop statistical tests for different types of Rényi functions;
- Prove that the theoretical results and the formulae for the Rényi functions are also valid for the values of q outside the interval $[1, 2]$;
- Study other models based on vector random fields (similar to Model 6), where the Rényi functions can be computed explicitly;
- Develop some approaches to study rates of convergence for the obtained asymptotics, that would serve as analogous of classical convergence rates in central and non-central limit theorems;
- Investigate changes of the Rényi functions depending on evolutions of random fields driven by SPDEs on the sphere;
- Apply the developed models and methodology to other spherical data, in particular, to new high-resolution CMB data from future CMB-S4 surveys which will be collecting $3D$ observations.

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Thank you